Some Variations of Domination and Applications.

Domination in Graph Theory is a widely studied area, in which there are more than thousand and five hundred papers published all over the world. There are different varieties of domination, which have a lot of applications in solving problems in our daily life. In this thesis we mainly study some variations of eternal domination and Roman domination in graphs.

This thesis contains seven chapters. First chapter contains basic definitions and preliminary materials needed for subsequent chapters. In chapter two, we introduce the concept of connected eternal domination. Eternal domination was introduced by A. P. Burger and his coauthors in the year 2004. If there exists a sequence of dominating sets $D = D_1, D_2, \ldots$ corresponding to a sequence of vertices $r_1, r_2, \ldots \in V$, satisfying the condition, $D_{i+1} = (D_i - \{w_i\}) \cup \{r_i\}$, where $w_i \in N[r_i] \cap D_i$ for $i = 1, 2, 3, ...$, then the set D_1 is called an *eternal 1 - secure set* or an *eternal dominating set* of the graph. The cardinality of a smallest eternal 1 - secure set in a graph G is called the eternal 1 - security number of G. It is denoted by $\sigma_1(G)$. A generalization of eternal 1 - secure set is obtained when we allow m guards to move in response to an attack. Corresponding dominating set is called an *eternal m - secure set.* The cardinality of a smallest eternal m - secure set in a graph G is called the eternal m - security number of G, which is denoted by $\sigma_m(G)$.

Sampathkumar and Walikar introduced the concept of *connected dominating set* in graphs. Connected dominating set is a dominating set S such that the induced subgraph $\langle S \rangle$ is connected. Connected eternal domination combines these two concepts.

The main result in this chapter states that the set of all simple connected graphs can be partitioned into two classes. One, the set of all graphs for which $\sigma_{cm}(G) = \gamma_c(G)$ and the other is the set of all graphs for which $\sigma_{cm}(G) = \gamma_c(G) + 1$. Here $\sigma_{cm}(G)$ is the minimum cardinality of a connected eternal dominating set and $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G.

In the third chapter we study a variation of eternal 1 - secure sets named safe eternal 1 - secure sets. A vertex is safe with respect to an eternal 1 - secure set S, if it is adjacent to exactly one vertex in S. We characterize the graphs having safe eternal 1 - secure sets, which has the property that all vertices - excluding those in the safe 1 - secure set - are safe. Also we introduce a new kind of directed graphs, which represent the transformation from one safe 1 - secure set to another safe 1 - secure set of a given graph and initiate a study of its properties.

Fourth chapter contains a generalization of safe 1 - secure set, which is obtained by removing m vertices from D_i and m other vertices are included to obtain D_{i+1} for all sets in the sequence $D_1, D_2, \ldots, D_k, \ldots$ of dominating sets of G. The number of vertices in a smallest safe eternal m - secure set of a graph G is the safe eternal m - security number and it is denoted by $\sigma_{sm}(G)$. We initiate a study on safe eternal m - security in graphs and give bounds for safe eternal m - security number. We also find the number for some classes of graphs and give characterizations in certain cases.

In the fifth chapter, we deal with the directed graph version of Roman domination. Let $D = (V, A)$ be a finite and simple digraph. A Roman dominating function (RDF) on a digraph D is a labeling $f : V(D) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has an in-neighbour with label 2. The weight of an RDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The Roman domination number of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF of D. We find Roman domination number of some classes of directed graphs.

In the sixth chapter, we initiate a study on linear relaxation of Roman dominating function (RDF) on a graph G. It is a function (labeling of vertices) $f : V(D) \rightarrow [0, 2]$, such that every vertex with label less than 1 has an in - neighbour with label greater than 1. Let u be a vertex having $f(u) < 1$ and v be its in - neighbour. Then the function values must satisfy the condition $f(v) - 1 \geq 1 - f(u)$. The weight of an RDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The fractional Roman domination number of a graph G, which is denoted by $\gamma_{R_f}(G)$, equals the minimum weight of an RDF of G. Many properties of the linear relaxation are discussed.

In the seventh chapter we explore the possibility of extending Roman domination to vertex - edge and edge - vertex versions of domination. For a graph $G = (V, E)$, a vertex $u \in V(G)$, ve-dominates an edge $vw \in E(G)$ if

- $u = v$ or $u = w$ (u is incident to vw), or
- uv or uw is an edge in G (u is incident to an edge that is adjacent to vw).

A set $S \subseteq V(G)$ is a vertex-edge dominating set (or simply a ve-dominating set) if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e.

For a graph $G = (V, E)$, an edge $e = uv \in E(G)$, ev-dominates a vertex $w \in V(G)$ if

- $u = w$ or $v = w$ (w is incident to e), or

- uw or vw is an edge in G (w is adjacent to u or v).

A set $S \subseteq E(G)$ is an edge-vertex dominating set (or simply an ev-dominating set) if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that e dominates v.

A vertex - edge Roman dominating function (VERDF) is a function $f : V(D) \rightarrow \{0, 1, 2\}$ such that each edge $e = uv$ is either incident with a vertex having function value at least one or there exists a vertex w such that either $wu \in E$ or $wv \in E$ and $f(w) = 2$. The weight of a VERDF f, $\omega(f) = \sum_{v \in V} f(v)$. The vertex - edge Roman domination number of a graph G (VERDN), which is denoted by $\gamma_{VER}(G)$, equals the minimum weight of all VERDF on G. In a similar way we can define edge - vertex Roman dominating function (EVRDF) and edge - vertex Roman domination number (EVRDN). These parameters are determined for many classes of graphs.

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